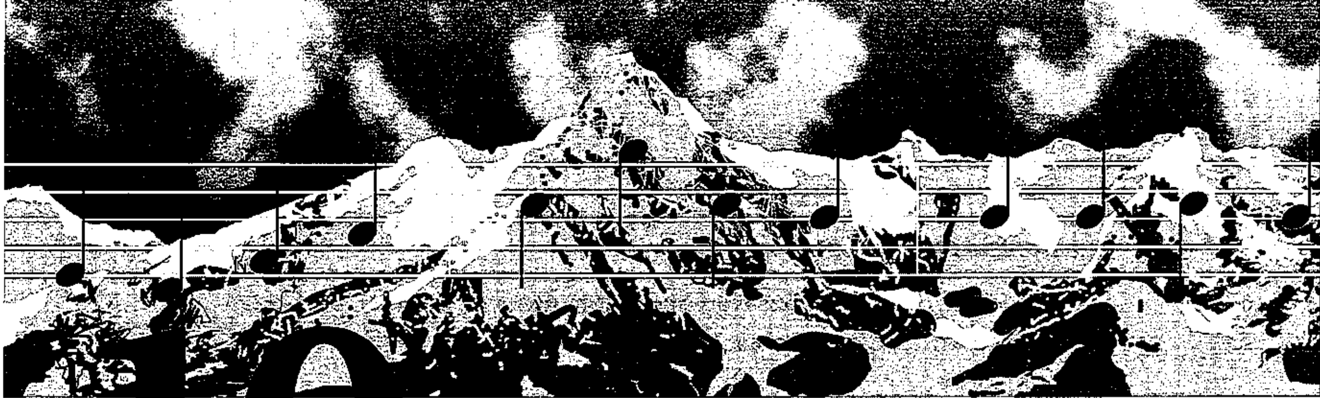


10

Scaling the Scale, Part I

Natural Vibrations and Pythagorean Tuning

TRACKS 77–88

Scaling the Scale, Part I leads students through an exploration of how the Pythagorean tuning for the Western major scale is derived from the natural vibrating harmonic frequencies in nature. Students begin by listening to various modes of standing-wave vibrations on a guitar string and seeing a diagram of their vibrating patterns. In a listening exercise, students hear the vibration frequency of each mode in relation to the fundamental and determine the corresponding note in the major scale. The musical distances between the various harmonic tones are defined and named as musical intervals. Students then establish the frequency ratios for the intervals between four of the harmonic tones. Finally, they use the frequency ratios for a fifth and an octave to calculate the frequency ratios for all seven tones of the major scale. This is the method used by Pythagoras to create the tones of the major scale in the sixth century B.C.

Scaling the Scale, Part II will lead students to discover an idiosyncrasy in the Pythagorean tuning. They solve the problem as Marin Mersenne did more than three hundred fifty years ago, using a geometric sequence to create the even-tempered scale—the scale used in modern fixed-pitch instruments.

Mathematics topics

Ratio, operations on fractions (multiplication, division, powers), problem solving, the work of Pythagoras. *Prerequisites:* Fraction multiplication and division, ratios.

Music topics

Scales, intervals, pitch recognition, frequency of pitch, harmonic series, timbre of musical instruments, piano keyboard. *No prerequisites.*

Use with the primary curriculum

- To provide a historical perspective
  When the work of Pythagoras is being studied, use Scaling the Scale to provide an example of the diversity and scope of this mathematician's work.
- To show a connection to physics
  Use Scaling the Scale to provide a graceful and authentic integration point between the physics of sound and mathematics.
- To review fractions
  Use Scaling the Scale to review operations on fractions.

Objectives

- To apply and reinforce fraction concepts
  Nonroutine problems can reveal gaps in basic skills and weaknesses in understanding. This context provides a rich, memorable experience to enhance retention and understanding of fractions.
- To use mathematics as a problem-solving tool in a nonroutine context
  Problem-solving creativity is enhanced with practice in nonroutine problem contexts.
- To see the mathematical and physical aspects of music
  The building blocks of music are derived from natural laws, and these natural laws adhere to an elegant and simple mathematical structure.
- To realize that Pythagoras concerned himself with more than just right triangles

Student handouts

- Natural Vibrations (reading; one per student)
- What to Do (resource page; one per group)

- Standing-Wave Vibrations (worksheet; one per student)
- Notes As Vibrations (worksheet; one per pair)

Materials

- CD tracks 77–88
- Ten-foot length of rope or heavy string (optional)

Instructional time

50–90 minutes

Instructional format

Scaling the Scale engages students in a wide range of topics in a relatively condensed activity. For this design to work, the music aspects have been simplified to the essentials. Take some time to assess the musical background of your class and note the discussion points in the teacher script that indicate where these simplifications take place. You may want to adjust the activity to include discussion of the music topics. Some of the beginning exercises may seem too simplistic for some of your students. Adjust your pace accordingly.

You will use the CD tracks as you lead the first half of the activity and help students understand some music concepts, terms, definitions, and basic mathematical relationships. (A summary of these concepts is in Step 1 from Scaling the Scale, Part II on pages 161–163.) Halfway through the activity, the format shifts to student groups working independently. Take the liberty of supplying instruction during this part of the activity, depending on the needs of your particular class.

Student preparation

Assign or read Natural Vibrations and hold a brief discussion emphasizing the idea that the laws of nature determine the special ways that things vibrate.

ACTIVITY SCRIPT

STEP 1 The natural modes of vibration of a guitar string

TRACK 77

The activity begins with an examination of the natural vibrations of a stretched guitar string. Students will examine a variety of different natural modes of vibration for the string. It is important to note that all the modes they will study occur on a string of constant length and tension. The only changes in sound result from the way the string is vibrating.

The first mode is the simplest: The string is plucked with no fingering or intervention of any kind, and a tone sounds. CD track 77 is an example of this. This note is called the *fundamental* or *first harmonic*, because it is created from the primary, most basic way a string can vibrate. The open guitar string vibrates back and forth for its full length, as shown on the diagram on the Standing-Wave Vibrations worksheet. The guitar string heard on the recording is vibrating at 130 Hz. It is called a *C note*. This activity will determine the frequencies of five different modes of vibration. Students should record on their worksheets the frequency of each mode both in terms of its actual value as heard on the CD and as multiples of the starting frequency, *f*.

Students see the first frequency values filled in on the student worksheet.

Steps 1 to 3 may be very basic for some students with music backgrounds. You can adjust accordingly how much time you spend developing and discussing the harmonics.

STEP 2 Another mode of vibration

TRACK 78

Explain that if a guitarist places a finger directly next to the string when it is plucked and instantly removes it, the obstruction of the finger forces the string to vibrate differently, pivoting around the center point as shown in the diagram on the Standing-Wave Vibrations worksheet. This creates the second harmonic. The pivot point, called a *node*, divides the string exactly in half. The node does not move, while the sections of string on either side vibrate back and forth in opposite directions.

Different modes of vibration can be effectively demonstrated by using a rope or a string. With one person holding each end of the rope, the first and second vibration patterns can be simulated. Simulating the third is almost impossible, but it can be fun for students to try.

Play CD track 78 and direct students' attention to the vibration diagram on the worksheet.

Ask students:

Can you describe the difference in sound between this note and the first harmonic? [One sounds higher than the other, maybe twice as high.]

What do you think is the relationship between the frequency of the first harmonic and that of the second harmonic? Refer to the hints on the resource page. [The second harmonic is vibrating twice as fast.]

Have students record these values in the appropriate space on their worksheets.

STEP 3 Frequencies of the remaining harmonics

TRACKS 79–81

Using the method you used with the first two harmonics, establish the frequencies for the third, fourth, and fifth harmonics. Play the CD track for each one (track 79 for the third harmonic, track 80 for the fourth harmonic, and track 81 for the fifth harmonic). Ask students to determine the frequencies and fill in their worksheets.

Strings vibrate in modes well beyond the fifth harmonic. All of the harmonics are referred to in music as the *harmonic series.* In mathematics, the term *harmonic series* refers to the nonconverging infinite sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots$$

The terms of the mathematical harmonic series are the same as the fractions representing the vibrating string length between nodes for each successive harmonic. The second harmonic is $\frac{1}{2}$ the vibrating length of the fundamental, the third is $\frac{1}{3}$ the vibrating length, and so forth. While musical string lengths in the musical harmonic series are not being added to create a sum, the mathematical counterpart does suggest the question of whether there are an infinite number of harmonics. [Theoretically, yes; in practice, no.]

STEP 4 Reflection on the harmonic series

TRACK 82

Ask students:

Do any of these notes sound like notes from music that you have heard?

Wind instruments as well as string instruments are subject to the same laws of nature as guitar strings. They create the same harmonic notes naturally. A bugle has no valves, so the length of its tubing is fixed in the same way that the guitar string's length remained fixed as it was placed in different modes of vibration. For a fixed length, the only notes that can sound are the notes of the harmonic series.

Recall bugle melodies that students may know, such as "Taps" or "Reveille." Ask a student volunteer to sing or hum them, or hum them yourself.

Some classic bugle tunes are played using harmonics on the guitar on track 82. Play the track.

The notes above the first harmonic are also referred to as *overtones.* Though not individually identifiable, many of these overtones above the fundamental frequency (the first harmonic) sound when an instrument plays a single note. In essence, the vibrating medium of an instrument actually vibrates in several modes simultaneously. The relative loudness of the overtones is what gives any note its characteristic sound—the way a trumpet, for example, can be distinguished from a guitar or a piano. The sound of an instrument is called its *timbre.* The relationship of the intensity of various overtones affects the timbre of a musical instrument. The timbre of a guitar string (the relative loudness of its various harmonics) can be changed by plucking the string at different places along its length, generating different combinations of vibration modes.

STEP 5 Harmonic tones and the major scale

TRACKS 83–88

This step of the activity will use six tracks of the CD. In a listening exercise, students will locate the notes created by the harmonic series in the major scale.

This exercise could be skipped to save time. You can give the positions of the notes as indicated on the teacher answer key directly to students.

To familiarize students with the major scale, explain that most music uses groups of notes called *scales.* The notes are given letter names and number names according to their position in the scale. Turn your attention to the piano diagram on the Standing-Wave Vibrations worksheet. Point out that the pattern on the keyboard repeats after seven notes and that the numbering reflects this. The eighth tone of the first pattern (octave) is the first tone of the next pattern. Play track 83 and ask students to follow along on the keyboard as the scale is played and the note numbers are announced on the CD. This activity should familiarize nonmusicians with the idea of scales.

This activity will consider only the white keys, the *diatonic tones* of the C-major scale. The black keys are notes that fall between the white keys. They create half steps and combine with the white keys to make up the *chromatic scale.* Tell students to ignore the black keys for now. They will consider them in Scaling the Scale, Part II.

Once students are familiar with the major scale, their next task is to learn to locate the harmonic tones.

Tracks 84–88 on the CD focus on each harmonic again, but in relation to the major scale. Each track sounds a different harmonic together with the C-major scale. Have students follow along on the piano keyboard as the scale sounds and place an X below the key that has the same pitch as the harmonic that is sounding. They should place the frequency for the harmonic below the X (in terms of the general value, f) at this time as well. Work your way through the tracks, pausing after each one to briefly discuss what was heard. The first harmonic is on track 84, the second is on track 85, and so on to the fifth harmonic on track 88.

Some students with musical experience will be able to locate the harmonics on the scale without the CD tracks. Others may struggle with what they hear on these tracks. Don't let them get bogged down. Coach students and, as a class, come to a conclusion as to the correct placements.

STEP 6 Musical intervals between harmonic tones

At this point you will teach the concept of a musical interval.

The activity may be successful with minimal discussion of intervals. You can make this decision based on your own comfort level, the amount of time you have, and the background of your students. The minimal working understanding of intervals for this activity is presented on the student resource page.

Explain to students that a musical interval is related to the distance between two notes or piano keys. (Any reference to a piano key is to white keys only. Remind students that they are ignoring the black keys for now.) The naming of the intervals is very logical. The name for the interval between tone number 1 and tone number 8 is an *octave* (from the Latin *octava,* meaning "eighth part"). The name for the interval between tone number 1 and tone number 5 is a *fifth.* The interval between tone number 1 and tone number 4 is a *fourth,* and so on. This pattern is true for all white keys on the piano. If you choose any white key and call it number 1, the interval created by going one white key up or down is a *second;* two white keys up or down, a *third;* and so on. The resource page simplifies this idea in a way that is functional for the activity.

This system may cause some confusion when you consider an interval as a distance. In a musical interval, a second is a distance of 1, a third is a distance of 2, and so on. This situation can be generalized as

$$\text{musical interval name} = \text{mathematical distance} + 1$$

As with many musical topics in this book, this treatment of musical intervals has been simplified to streamline the activity for nonmusicians. The interval names second, third, fourth, and so on are general terms. Each can be designated more specifically as a minor/major second, minor/major third, perfect or augmented fourth, and so on, each term representing a different musical distance. If you examine the piano keyboard, it is clear that there is no black key between notes 3 and 4 and notes 7 and 8. This suggests that the distance between notes 3 and 4 and the distance between notes 7 and 8 are smaller than the distance between the other white keys. This is true. However, using the general names is a simplified and accurate system of naming the intervals in relation to movement between any white keys. If your musical students bring this up, acknowledge it and explain that, for the purposes of the activity, this point is not essential. In fact, it may confuse the matter and be distracting to your goal. You should refer to the intervals by their general names. Technically, however, the intervals between the harmonic tones considered in this activity are the octave, perfect fifth, perfect fourth, and major third.

On their worksheets, have students label the musical intervals between the harmonic tones with the appropriate names. The appropriate names are shown on the teacher answer key.

STEP 7 Frequency ratios for musical intervals

This step develops ratios between the frequencies of the notes of an octave, fifth, fourth, and third (the first four overtones).

Help students establish a frequency ratio for each interval created between adjacent harmonic tones. Explain to them that the object is to find a factor that will allow them to calculate the frequency of a note a given interval above the given note. For the interval of a fifth, we can let the factor be x, so $2f$ times x equals $3f$. Solving for x gives a factor of $\frac{3}{2}$. To create the interval of a fifth, multiply a note's frequency by $\frac{3}{2}$ to get the frequency of the higher note.

Students will indicate these ratios on the chart on the Standing-Wave Vibrations worksheet. As you teach this step, keep in mind that a musical interval name is the mathematical distance plus 1.

When you establish the ratios for the musical intervals from the harmonic tones, it is important to keep in mind that the ratios apply for those intervals (distances) wherever they may occur on the piano keyboard.

STEP 8 Adding intervals and creating the major scale

Give student groups the second worksheet, Notes As Vibrations. As students work, you can move from group to group, coaching and clarifying when necessary. Alternatively, you can complete the worksheet as a class.

To get groups started, clarify that laying intervals end on end should be done in such a way that the top note of the first interval is the bottom note of the second interval. That is, the intervals share a common note.

Students may need some assistance in finding a process by which to calculate the ratios of the scale tones. The instructions are left purposely vague to give students a greater opportunity to explore and discover. Direct them to the hints on their worksheet and provide direct instruction when absolutely necessary.

If students have begun working on the development of the scale (the frequency ratio to fundamental), they can finish the chart for homework. It is important for students to be sure their answers to the first problems are correct, since each problem is dependent on the preceding answer.

STEP 9 Closure: Ratios for Pythagorean tuning of the major scale

Bring the class together and check the ratios for each tone of the major scale. Explain that the Pythagorean tuning is only one of many that have been used through history. Early scale tunings, such as the Pythagorean tuning, presented problems for musicians as music became more sophisticated and different keys were used. In Pythagorean tuning, musical intervals of the same name (such as seconds or whole steps) that occur at different places in the scale actually have different ratios. This idiosyncrasy was finally solved with the tuning method currently used as a standard for Western music, called *even temperament.* Even temperament is presented in the activity Scaling the Scale, Part II.

Notice that the frequency ratio for the third created by the harmonic tones is $\frac{5}{4}$, while the frequency ratio for the third calculated in the activity is $\frac{81}{64}$. This is one example of the idiosyncrasies of Pythagorean tuning. The third from the harmonics is a pure tuning, while the third from the activity is the Pythagorean tuning. The ratio for the pure third is actually smaller than for the Pythagorean third. Many musicians contend that the pure third (natural) is more pleasing than the Pythagorean or tempered third (calculated). Another type of tuning, called *just intonation,* substitutes the ratio for the pure third into the Pythagorean tuning. The debate over pure tunings versus mathematically altered tunings continues to this day among some musicians.

FOLLOW-UP ACTIVITIES

Scaling the Scale, Part II

Scaling the Scale, Part II reveals an idiosyncrasy of stacking intervals and multiplying frequency ratios. Chromatic notes are introduced, and students solve the centuries-old problem of tuning the scale by creating the system currently used in Western music: even temperament.

Writing prompts

- What did you learn in today's activity?
- What part of music is a human invention, and what part is determined by nature?
- Is it surprising to you that Pythagoras was so involved in solving musical problems? Why or why not?

Textbook assignments

- Textbook problems that apply operations with fractions, arithmetically or algebraically
- Textbook problems that use the Pythagorean theorem

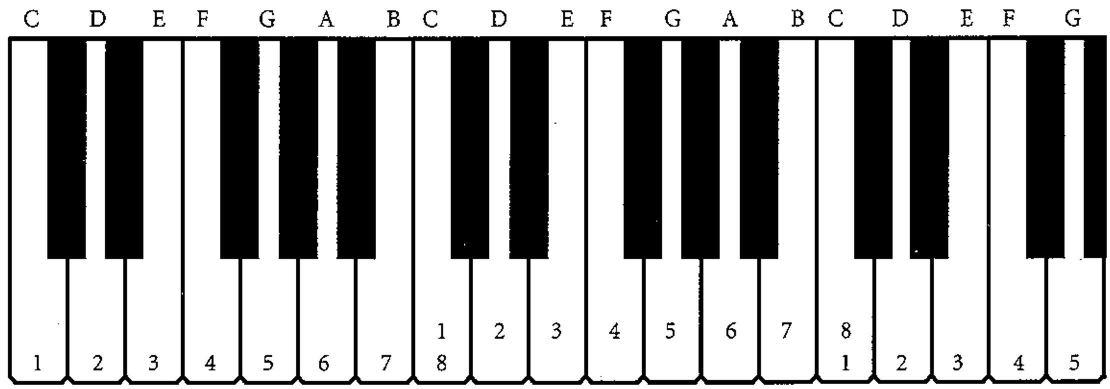
Extension

Challenge students to find ratios for the scale tones by applying the same interval stacking process using the ratios of the fourth and the third of the harmonic tones in addition to those of the fifth and the octave. The pure-scale tuning can be developed using these ratios.

ANSWERS

Standing-Wave Vibrations

	Frequency	
	Example	**General**
First harmonic	130	f
Second harmonic	260	$2f$
Third harmonic	390	$3f$
Fourth harmonic	520	$4f$
Fifth harmonic	650	$5f$



	X		X		X		X		X
Harmonic frequency	f		$2f$		$3f$		$4f$		$5f$
Musical interval		Octave		Fifth		Fourth		Third	
Ratio for interval		$\frac{2}{1}$		$\frac{3}{2}$		$\frac{4}{3}$		$\frac{5}{4}$	

Notes As Vibrations

1. A fifth and a fourth
2. $\frac{3}{2}$ and $\frac{4}{3}$
3. $\frac{3}{2}$ times $\frac{4}{3}$ equals $\frac{2}{1}$
4.

Musical interval	Frequency ratio to fundamental
The fundamental	1
The second *up 2 fifths, down 1 octave* $\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{9}{8}$	$\frac{9}{8}$
The third *up 4 fifths, down 2 octaves* $\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{3}{2}\right)^4\left(\frac{1}{2}\right)^2 = \left(\frac{81}{64}\right)$	$\frac{81}{64}$
The fourth, frequency of the natural harmonic	$\frac{4}{3}$
The fifth, frequency of the natural harmonic	$\frac{3}{2}$
The sixth *up 3 fifths, down 1 octave* $\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{3}{2}\right)^3\left(\frac{1}{2}\right) = \left(\frac{27}{16}\right)$	$\frac{27}{16}$
The seventh *up 5 fifths, down 2 octaves* $\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{3}{2}\right)^5\left(\frac{1}{2}\right)^2 = \left(\frac{243}{128}\right)$	$\frac{243}{128}$