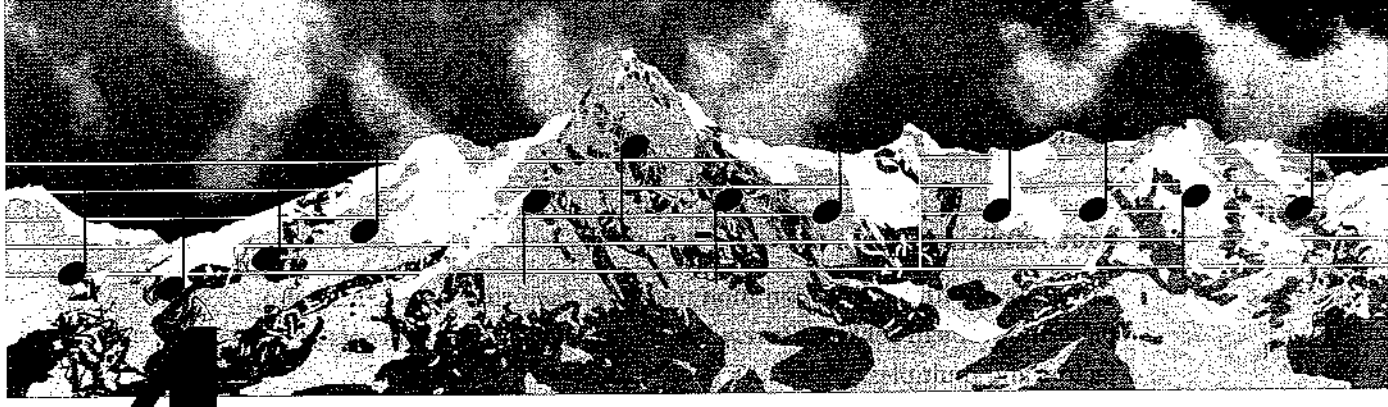


4

The Multiples of Drummers

TRACKS 23–28

The Mathematics of Polyrhythms

This fun activity challenges students' motor coordination and auditory perception and helps them discover a rule for finding the least common multiple (LCM) of a set of numbers. Student groups clap various rhythmic cycles simultaneously and represent them symbolically on a polyrhythm chart. Polyrhythms created from the superimposed rhythms are performed, listened to at various tempos, and displayed on the chart to show the relationships between factors and multiples that lead to a general rule for finding the LCM of any set of numbers. After the performances, a variety of applied music problems involving polyrhythms provide multicultural perspectives and present some powerful extension problems and explorations.

The performance nature of the activity provides a different access point for the review of multiples and the LCM. The entire class is engaged in an interactive performance, physically experiencing shifting resonance points between different clapping patterns. Regardless of students' mathematical backgrounds, The Multiples of Drummers provides a fresh view of multiples and a unique experience.

Mathematics topics

Counting, multiples, least common multiples, factoring, exponents, ratio, patterns, problem solving. *Prerequisites:* Knowledge of the concepts of prime factors and multiples.

Music topics

Beat, rhythm, polyrhythms, phrasing, tempo, accents, performance.
No prerequisites.

Use with the primary curriculum

- To introduce the LCM for prealgebra
  This activity can provide a way for students to develop the rule for the LCM in a prealgebra course.
- To review the LCM for algebra I
  Use this activity before lessons on the LCM of algebraic expressions or on adding and subtracting algebraic fractions. Solid understanding of and skill with LCMs are fundamental to efficient manipulation of algebraic fractions, and a weakness in this area can undermine a student's success well into calculus courses.
- As a special project
  The Multiples of Drummers can be used to set the context for the extensions as special projects or problems of the week.
- For review, enrichment, or assessment
  The Multiples of Drummers can provide a powerful assessment or enrichment tool after the concepts of multiples and the LCM have been taught using another curriculum.

Objectives

- To enhance understanding of multiples and factors
  Practice with using the relationships of multiples in a unique context can improve understanding for all students.
- To strengthen retention of concepts
  When students discover mathematical rules and concepts through observation and active experimentation, they remember them longer.
- To provide access to mathematical concepts for students of various learning styles
  The combination of kinesthetic, visual, and aural representations of quantitative relationships can provide access for learners with strengths in these areas.

- To foster multicultural insights
  Mathematics is used as a tool to help students gain new appreciation and understanding of the art of different cultures.

Student handouts

- The World Is a Drummer (reading; one per student)
- Help and Information (resource page; one per pair)
- Polyrhythm Chart and Problems (worksheet; one per student)
- Polyrhythms in Music (worksheet; one per student)

Materials

- CD tracks 23–28
- Metronome (optional)

Instructional time

30–60 minutes

Instructional format

This activity can give you an opportunity to fulfill a possible fantasy—to conduct a 20-piece music ensemble. Conductor will be your primary role. To begin, divide the class into three large groups. These groups will perform different rhythms by clapping to a metronome beat. The entire class works together to perform and analyze an assortment of rhythms. Use an overhead transparency of the polyrhythm chart, and mark rhythms during the whole-class analysis of the performances.

Following the whole-class performances, student pairs use the resource page to work through a set of problems guided by your coaching. Periodically the entire class comes together for debriefing to ensure that all students are reaching accurate conclusions. The final set of problems, Polyrhythms in Music, can be completed by students in pairs, with little assistance from you—or it can be assigned as homework.

Remember to review the content of the resource page so that you can adjust the amount of direct instruction you deliver to match your teaching style, the extent to which you intend to create a constructivist experience for your students, and the level of independence your students can handle.

Student preparation

As is the case with all the activities in the book, it is important to set the context for, and warm students up to, the type of activity they are going to be doing. Have them read The World Is a Drummer either the night before or in class

prior to the lesson. If your class has been doing several other music activities and you are short on time, you could skip this reading.

ACTIVITY SCRIPT

STEP 1 Introduction and background information

Review and discuss the reading and give students an overview of the activity. Explain that they will clap rhythms and chart their performances to discover the mathematics of rhythm. Don't forget to emphasize that they should have fun in the process!

At this point introduce the following basic musical concepts and terms:

Musicians perform music in reference to a steady pulse called the *beat.* In musical practice the beat is played in some form by a musician, directed by a conductor, or just imagined in the musician's mind. In this activity it is represented by a steady metronome pulse. The speed of the beat is called the *tempo.* Patterns of accents on the beats create *rhythms.* Combining two or more different rhythms creates *polyrhythms.* We will be concerned with finding the phrase length of rhythms and polyrhythms, which is measured by the number of beats it takes for the rhythm to repeat its cycle.

Don't get bogged down with trying to ensure complete understanding of the background information. The meaning of these ideas will be clarified after the first performance.

The tempo of musical pieces is indicated in beats per minute to conductors and musicians. The performance tempo for this activity is 100 beats per minute.

STEP 2 Create groups and assign roles

Here are two options for student groupings and roles depending on the mathematical background of your students.

Option 1—prealgebra and algebra 1: Divide the class into three equal groups. Two of the groups will be clappers performing two different rhythm cycles simultaneously, and the third group will be observers. The observers will count beats and look for the beat numbers on which the two clapping groups clap together. The clapping groups will be referred to as group A and group B. As each new polyrhythm is performed, it is important to have students shift roles so that all students have equal time clapping. This can be time consuming, however, since having new groups learn parts for each performance can become tedious.

Option 2—algebra 1, algebra 2, and above: Divide the class in half to make two even clapping groups and eliminate the observer group. For more advanced students, the observer group is unnecessary—it will be obvious to most of these students even before the performance that the resonance point is the LCM of the two primary rhythms.

Be creative by suggesting that students use body movements or spoken words instead of clapping. To make the activity a visual and kinesthetic experience, have students in clapping groups raise both hands suddenly at a point where they otherwise would have clapped. Students could also shout a word at the same instant, or they could skip the hand movements altogether and use only words. These variations can be great fun for the class and very valuable for kinesthetic and visual learners.

STEP 3 Perform the first polyrhythm, 2:3

TRACK 23

The process described below will be repeated throughout the activity with different rhythms.

Explain that all the clapping will be done in reference to the metronome beat. Play the metronome on CD track 23 or use a metronome set at 100 beats per minute.

Group A will clap the first rhythm listed on the chart, 1:2. This ratio indicates one clap every two beats, so the group A clappers will be clapping on beats 0 (go), 2, 4, 6, and so on. Play the metronome beat and have the students practice this until they are clapping together; then stop and instruct group B.

Group B will clap the second rhythm listed on the chart, 1:3. This ratio indicates one clap every three beats, so group B clappers will clap on beats 0 (go), 3, 6, 9, and so on. As with the previous group, play the metronome and have students clap their rhythm until they are together and confident.

Groups A and B now clap their rhythms simultaneously. You will have to conduct this. Cuing students to start together can be a little tricky. The best way to do it is to let the metronome run for a few beats and then give a "one, two, go" instruction with students beginning their clapping on the word *go*. Giving this count on every other beat is the easiest to hear and comprehend for all involved. It may take a few attempts to coordinate the two groups.

If you are using observers, have them take note of what beat numbers groups A and B clap on together. Or have the clapping groups identify the beat number of the common claps. The distance between the simultaneous claps is the phrase length.

The rhythm made of two separate rhythms, 1:2 and 1:3, is called a *polyrhythm*. The resulting polyrhythm in this case is 2:3.

The polyrhythm 2:3 is a very common polyrhythm found in many types of music, from rock to classical to jazz. Playing a three-beat phrase in the space of a two-beat phrase is called a *triplet* in musical terminology. In many African cultures, it is believed that 3 is a male number and that multiples of 2 are female numbers. It is also believed that both male and female elements need to be present to reach perfection in any endeavor, leading to the presence of 2:3 polyrhythms in much of African music.

Students now fill in the polyrhythm chart for the rhythms they have performed. Have them place marks on the appropriate beat lines to signify claps. Using a different symbol for each polyrhythm can help students visually differentiate the polyrhythms. Confirm with the class that the phrase length for a 2:3 polyrhythm is 6. Consult the observers for verification, and direct all students to fill in the phrase length in Table 1.

Play the recorded demonstration on CD track 24. This demonstrates the 2:3 polyrhythm played perfectly and also at faster tempos. Notice that as the tempo increases, the rhythm sounds less awkward and almost familiar. You may want to bring in other obvious musical examples.

Students will determine the phrase length in various ways: from their mathematical concept of common multiples, from the observer group, and from the polyrhythm chart. If discussion reveals varying methods and perceptions, it can be valuable to have students share these with the class. Even for students who see the phrase length as obvious from the outset, this simple problem is a useful starting point because the problems get more difficult.

STEP 4 Perform the next polyrhythm, 3:4

Follow the process outlined in Step 3 to perform the 3:4 polyrhythm.

As previously suggested, you may choose to rotate the roles of the groups. If not, it is efficient to let the group clapping the 1:3 rhythm continue to do so and then have the other group switch from clapping 1:2 to clapping 1:4.

After students perform the 3:4 polyrhythm, play CD track 25 to demonstrate the sound of the 3:4 polyrhythm played perfectly. Do students think it sounds more musical as the tempo increases?

A good example of music with this polyrhythm is "Fantasy Impromptu" by Chopin, in which the right hand plays four beats to the left hand's three.

Have students fill out the polyrhythm chart for these rhythms, along with Table 1.

The polyrhythm chart closely models how pitch frequencies interact. Musical notes are created by pulses, like clapping, happening at very fast frequencies. Different pitches with frequencies that have common factors sound more harmonious to our ears than those with no common factors. An interesting parallel to this phenomenon exists with polyrhythms. Rhythms that fit well with each other seem to have phrase lengths with common factors. This idea is discussed in more detail in the extensions at the end of this activity script. The numerical relationships between pitch frequencies is the subject of Scaling the Scale, Part I.

STEP 5 Complete the remaining polyrhythm performances and polyrhythm problems

TRACKS 26–28

You can choose to perform as many of the remaining polyrhythm possibilities as you like. CD track 26 has polyrhythm 4:5, and CD track 27 has polyrhythm 2:3:5.

Feel free to experiment with some other combinations. For the purposes of the activity, the entire polyrhythm chart should be filled out along with Table 1 before you proceed to the polyrhythm problems. I recommend that students perform at least 4:5 and 2:3:5. For 2:3:5, you will need three clapping groups. Note that Table 1 contains the polyrhythm 3:5, which is not on the CD. Students will need to think about the mathematics rather than just recording what they observe.

When the chart is complete, pose the question of what it would sound like if all the rhythms were played simultaneously: polyrhythm 2:3:4:5:6:7. Intuition might suggest that this would sound like chaos, considering that there are few common factors and that the phrase length would be very long. If you really feel ambitious, you might try having students perform it. At least play CD track 28 to demonstrate 2:3:4:5:6:7. Do students think it sounds chaotic?

Some students will want to work out the phrase length of 2:3:4:5:6:7. This is fine, but put off presenting solutions or giving hints until problem 3 of Polyrhythms in Music is encountered. (The phrase length is 420 beats.)

Dance in many African cultures incorporates complex polyrhythms. A good dancer will define up to four different rhythms at once, one with each limb! On first glance, these dances may appear to consist of random movements, but careful observation reveals a high level of sophistication and order.

After some discussion with the class, play the CD track. It is surprisingly musical, and one could actually dance to it. It has the quality of African rhythms that use many polyrhythms simultaneously.

Pause to have students analyze why the sound of all these rhythms together has a sense of cohesion and why it sounds organized. For one thing, many of the rhythms within it have short phrase lengths, and the listener is drawn to those. The short lengths form a backbone for the structure. The phrases that take many beats to repeat are heard almost as colorful accents and do not distract from the organization of the shorter phrases.

STEP 6 Analysis: Polyrhythm problems

At this point, the activity mode shifts from performance to analysis. It can be effective to maintain the whole-class engagement and work through these problems in a discussion format as a class. Feel this out. Depending on the personality and background of your class, you may decide to work through these problems with students in pairs. The questions form a directed discovery sequence that ends with problem 7, which defines the general rule for determining the least common multiple of any set of numbers or algebraic expressions.

It is not expected that students will reach an exact rule on their own purely from the information obtained in the activity. The extent to which this rule is student-generated will vary from class to class and within a class. This sequence of problems needs to be supplemented with your gentle guidance and leading questions. Remind students to use the resource page for more help. Bring the class together after problems 5, 6, and 7 to share conclusions. Before moving on, be sure all students have an accurately articulated version of the general rule for finding the phrase length of any polyrhythm, this being the rule for finding the LCM for a set of numbers.

To help students discover the rule for finding the LCM, bring the class together and lead them through some more examples of finding the LCM for numbers that contain common factors. As students observe the answers for a set of problems, they can find a pattern to create the rule. First establish what the LCM means. Then, with the class, express each number and the LCM in prime factorization form using exponents, and look for patterns. Present as many examples of this as necessary. For example:

The LCM of {10, 15, 12} is 60.

Students factor these numbers into primes:

$10 = 2(5)$ $15 = 3(5)$ $12 = 2^2(3)$ $LCM = 2^2(3)(5)$

Compare the factors of the LCM with the factors in the set of numbers. Notice that the LCM is the product of the greatest power of each prime factor in the set of numbers.

STEP 7 Polyrhythms in Music

Students can work on their own to complete the applications in Polyrhythms in Music. Problem 4 presents a complex mathematical relationship that cannot be worked out using simple multiples. Suggest that students do the problem graphically, and ask them what is different about it. Notice that the answer, 22, can be determined mathematically by $2(5) + 3(4)$. While this suggests a general formula, it works only for particular sets of numbers. This problem is included to introduce another level of complexity and to provide a launching point for an extension problem.

FOLLOW-UP ACTIVITIES

Textbook assignments

It is important to follow The Multiples of Drummers with some skill practice that requires finding LCMs either numerically or algebraically. Adding and subtracting fractions or pure LCM exercises are given meaningful context by this activity, and practice will help solidify the pure mathematical skill and understanding offered by the activity.

Writing prompts

- What did you learn in today's activity?
- What music that you are familiar with uses some of the polyrhythms you clapped today in class?
- Which polyrhythms sounded the most familiar? The least familiar?
- Which polyrhythms were the easiest to perform? Explain why in mathematical terms.
- Which polyrhythms were the most difficult to perform? Explain why in mathematical terms.
- Did today's activity help you understand more about multiples? Why? How?

Extensions

- Connection to pitch and frequency
  Explore and research how the polyrhythm chart models the interaction of different pitch frequencies. When two different pitches sound simultaneously, a perceptible pulse (a dull throbbing) is heard, called the *beat frequency.* This frequency is the difference of the frequencies of the two pitches. It can be interesting to compare this frequency relationship to the

relationships in polyrhythms. Our activity considers the beat as a measure of time and observes repetitions of a rhythmic cycle (phrase) in relation to it. The frequency of repetition of a phrase can be expressed as the number of cycles (phrase repetitions) that occur per unit of time, in this case, cycles per beat. Thus the rhythms have frequencies of $\frac{1}{2}$, $\frac{1}{3}$, and so on. This prompts the following question: Is the relationship of the phrase length of a polyrhythm to the lengths of the two primary rhythms that create it analogous to the relationship of the beat frequency to the frequencies of the pitches that create it?

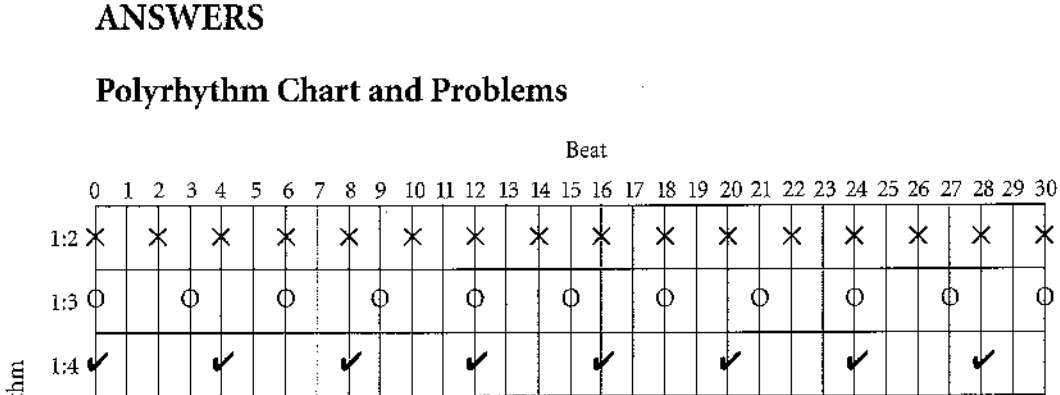
It turns out that the formula of subtracting the frequencies to obtain the beat frequency holds with polyrhythms in some cases and not in others. For example, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, and $\frac{1}{6}$ is the frequency for the polyrhythm created by 1:2 and 1:3. This formula also works for 1:4 and 1:6, rhythms with common-multiple lengths. It does not work for 1:3 and 1:5, however. This is a curious phenomenon, rich with possibilities for exploration and analysis.

- Patterns for operations on fractions in the polyrhythm chart
  If you study the polyrhythm chart, you can see many interesting patterns that the different polyrhythms make with each other. See if you can find a simple way to add fractions based on the chart. Of course you can just add the fractions mathematically with paper and pencil, but there is a way to get the answer directly from the chart. One system works only for fractions with 1 in the numerator. For example, to add $\frac{1}{2} + \frac{1}{3}$ the denominator of the sum is clearly the phrase length of the polyrhythm created by 1:2 and 1:3 (the LCM of 2 and 3). The numerator of the sum is found by adding the phrase lengths of the two rhythms, 2 + 3, so $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Does this system work for fractions with a number other than 1 in the numerator? Why or why not? Could you find a system in the chart that would work for all fractions? What would it be?
- Class performance
  Have a student bring in a drum machine so that you can play any combination of rhythms at will to find a correlation between the sound and the mathematical relationship.
- A brick-laying problem
  Problem 4 of Polyrhythms in Music is similar to a classic brick-laying problem: Two layers of bricks are laid, with each layer constructed from bricks that alternate in length. What is the minimum length the wall can be using specified brick lengths if the brick layers must match perfectly at the end of the wall?

Generalize the problem for alternating lengths in the first layer of a and b and in the second layer of c and d. As mentioned earlier, for some cases the solution is $ad + bc$.
Extension: Establish a rule for all cases, or determine inequalities for a, b, c, and d where particular formulas will work.

ANSWERS

Polyrhythm Chart and Problems



1.

Table 1	
Polyrhythm	**Phrase length**
2:3	6
3:4	12
3:5	15
2:3:5	30

2. In these cases, the phrase length of the polyrhythm is the product of the phrase lengths of the individual rhythms.

3.

Table 2	
Polyrhythm	**Phrase length**
2:4	4
2:6	6
2:6	6
2:3:4	12

4. No; the phrase length is less than the product of the individual rhythms. On the polyrhythm chart the first rhythm is a divisor of the last.
5. Polyrhythm numbers in Table 1 are not divisible by each other and have no common factors other than 1. In Table 2, the last number is divisible by the first.
6. The phrase length of the polyrhythm is the least common multiple (LCM) of the phrase lengths of the individual rhythms.
7. You have to find the LCM of the phrase lengths of the individual rhythms that make up the polyrhythm. For any group of numbers or expressions, the LCM is the product of the greatest powers of the prime factors of the numbers or expressions.

Polyrhythms in Music

1. The LCM of 13 and 15 is 195.
2. The LCM of 2, 5, and 4 is 20.
3. The LCM of 2, 3, 4, 5, 6, and 7 is 420.
4. 22 beats. Use a graphic approach. Mark segment lengths, alternating 2 and 3 on one line and 4 and 5 on another line. They match on the twenty-second beat.