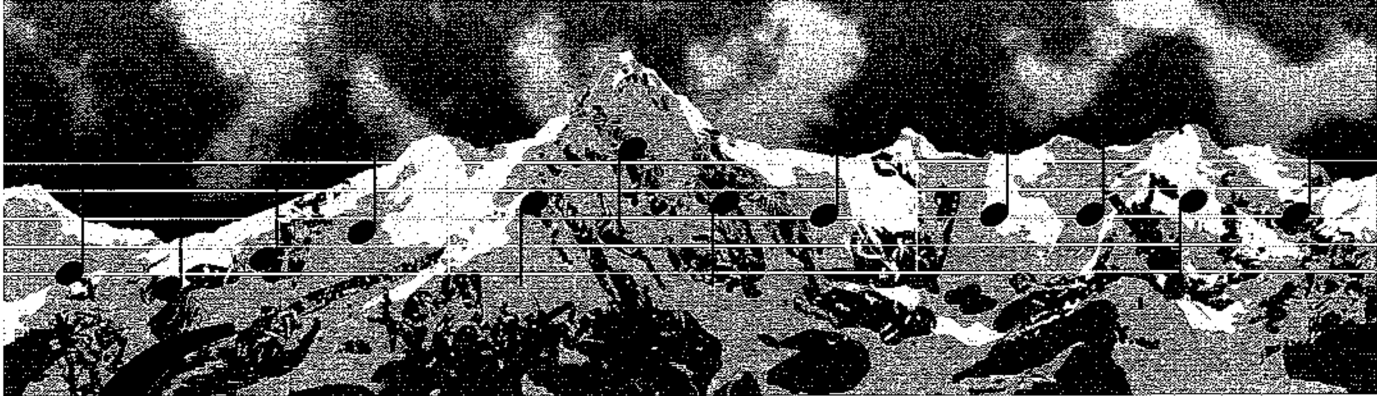


6

Functional Composer, First Movement

TRACKS 35–40

A Mathematical Solution to Writer's Block

In Functional Composer, First Movement, students use transformations of an algebraic function to solve a music composer's problem. They represent a musical motif as a function, calculate the function's transformations, draw and see the graph of those transformations, and hear the transformations played as melodic lines. The qualities that relate the transformations of the function to one another mathematically are present in their melodic counterparts, so students experience these mathematical relationships sonically, in addition to numerically and visually. They discover that these relations are the glue of a musical composition that has structural integrity and hence sounds good and makes sense aesthetically. The activity closes with students identifying the melodic counterparts of the transformations in a musical composition. The composer's problem is solved.

TEACHER NOTES

Mathematics topics

Definition of a function, function notation, graphing transformations of functions (stretching, shrinking, reflecting, translating), composite functions, order of operations, in/out tables. *Prerequisites:* Ability to graph ordered pairs of numbers on a Cartesian coordinate system using in/out tables.

Music topics

Musical notation (pitch), musical composition and melodic structure, ear training, transposition, modes. *No prerequisites.*

Use with the primary curriculum

- To introduce the definition of a function
  Use Functional Composer to introduce the notion of a function as a pairing of integer sets and to provide practice using in/out tables to graph on a Cartesian coordinate system.
- To introduce transformations of functions and their graphs
  Use Functional Composer to present the idea of transforming a function when interpreting and creating graphs of conic sections or logarithmic, trigonometric, or algebraic functions.
- To review functions and transformations
  After the idea of function or transformations has been taught, use Functional Composer to show an application and enrich student understanding.
- To illustrate transformations in a geometry class
  Functional Composer can reinforce student understanding of mappings, dilations, and reflections.

Objectives

- To understand functions and their graphs
  The experience of hearing, calculating, and creating visual representations enhances understanding for all students and gives audio and visual learners a chance to shine.
- To retain the concept of a function
  Retention will be increased by the very memorable experiences of Functional Composer, First Movement.
- To observe unity, elegance, and beauty
  Seeing an artistic problem solved with mathematics stimulates questions about how mathematics relates to nonmathematical realms of intuition and inspiration.

Student handouts

- A Composer's Problem (reading; one per student)
- What to Do (resource page; one per group)
- Melodic Variations and Transformations (worksheet; one per pair)
- Melodic-Contour Graph (worksheet; one per pair)

Materials

- CD tracks 35–40
- Overhead transparencies of student worksheets

Instructional time

50–60 minutes

Instructional format

You conduct the activity by using the CD tracks (pausing and repeating tracks when necessary) together with overhead displays of the transformations. Students work in pairs. One student in each pair uses the Melodic Variations and Transformations worksheet to calculate the function transformation values (using in/out tables) and to write corresponding melodies. The other student uses the Melodic-Contour Graph to graph the various melodic variations (function transformations). It is a good idea to ask students to make notes after each transformation to summarize the nature of the transformation. Try to have your students supply the conclusions stated in the activity script. This process is continued in Functional Composer, Second Movement. Students can make a table in their notes similar to the chart for that activity.

The melodic variations as played on the CD are given a rhythmic shape to enhance their appeal. This rhythmic element has been excluded from the musical notation of the motif to focus attention on the melodic/function aspects.

Instead of using the prerecorded CD, ask a student to bring a musical instrument to class to play the melodies developed in the lesson. You will still want to use the CD to play the final composition created from the transformations. Musically inclined students can greatly enhance the engagement of all students in class discussion, and even allow students to create their own compositions from the transformations.

Student preparation

To save time on the day of the activity, start the day before with A Composer's Problem (read by students or read aloud by you). Discuss the reading and ask musicians in the class to comment on their own experience.

ACTIVITY SCRIPT

STEP 1 Establish the context

Set the context with students by referring to the reading or describing a hypothetical scenario of a composer who is experiencing writer's block.

> A composer is trying to write a piece of music. She has been inspired with a little melody (motif) but can't seem to come up with any more music that sounds good or makes sense with the melody. Everything she tries to add sounds unrelated to the original melody. She is very frustrated; she is experiencing a bad case of writer's block.

Then suggest that there is a mathematical solution to her problem.

It can be fun to dramatize the opening scenario and ask students to share their own experiences with writer's block and why it happens. Students may have experienced blocks in other art forms, such as painting or poetry.

The process used in Functional Composer, First Movement has been used consciously and unconsciously in composition and improvisation by musicians and composers from Johann Sebastian Bach to John Coltrane. Bach's fugue and canon forms were assembled from vast assortments of melodic variations similar to the function transformations we will develop. Jazz improvisers invest great effort in creating *riff patterns* that they use as resources for improvisation. Many of these patterns are generated using a process identical to mathematical transformations.

STEP 2 Establish the number/note system

Turn students' attention to the number/note reference key on the Melodic Variations and Transformations worksheet. Explain that musical notes are different frequencies of vibration that our ear hears as different *pitches*. You can mention that musicians name the different pitches with letters and that the pitches are arranged in a *scale*. The starting point for the musical scale is a note called the *key note* or *tonic* of the scale.

Students will give each note on the scale a number, letting the key note be number 1. The number/note reference for the scale is like a number line. High notes (fast vibrations) are named with higher numbers, and low notes (slow vibrations) are named with lower numbers.

Point out that musicians represent the different pitches by using dots on a set of lines and spaces called a *staff*. Each position of a dot on the staff corresponds to a different pitch. Take time to examine the number/note reference key and point out that the dots fall on either a line or a space. For Functional Composer, First Movement, students need only make a connection between the number and the position of the note on the staff.

Use your own judgment as to how many musical terms you discuss. It is not essential that students know these terms for the success of Functional Composer, First Movement, and too much information can be a distraction. If you do decide to use many musical terms, you might want to keep a word list of new terminology. The letter names of notes need not be mentioned, as this can distract from the goal of the exercise—that students make the geometric observation of where each number's note is positioned on the staff.

STEP 3 Introduce the original motif: $y = f(x)$

TRACK 35

Play the recording of the composer's original motif, and show students that it consists of the following sequence of notes: 3, 5, 2, 4, 3, 0, 1. Play this melody several times if you like.

If your students are musically inclined, you might ask them to determine the numerical values of the notes solely from listening to the melody. You could even ask students to sing the numbers with the melody.

The number values for the notes of the melody will be the function values of $f(x)$. The input values, x, are the numbers of the notes in sequential order: 1 for the first note, 2 for the second, and so on.

Discuss how this fits the definition of a function: the pairing of numbers from two sets, an input value x and an output value $f(x)$. Our motif function is defined as the pairing of numbers between the sequential value of the term, x, and the numerical value of the note, $f(x)$. Students may be uncomfortable because no mathematical rule relates the input values to the output values. A mathematical rule is not essential in the definition of a function, and working with this idea can actually help broaden students' understanding of a function.

It is intriguing to entertain the idea of having no formula to describe the relationship between input and output values. Is this relationship outside the realm of mathematics? What determines it?

One student from each pair should write the first in/out table for this melody and place the correct notes on the staff. Refer to the completed Melodic Variations and Transformations worksheet and Melodic-Contour Graph. (Note that students will complete these worksheets one function transformation at a time.) The other student in each pair will draw the melodic contour on the Melodic-Contour Graph. This motif is the starting point; students will now calculate an assortment of transformations, plot them, and listen to them.

Discuss with students whether connecting the points with line segments on the contour graph is an accurate representation of the function. Have them explain why it is not. Assist them, if necessary, by explaining that the notes do not gradually change from one to the next; instead, the change from one note to the next is like a step. Then point out that they will connect the dots to help them gain a visual image of the shape of the functions/melodies.

To build students' group-work skills, you can have them switch roles for each transformation. This can increase students' interaction but will require more time, since students need to learn both tasks—ideally from each other.

STEP 4 Introduce the first transformation: $y = f(x) + 2$

TRACK 36

Ask students for suggestions of how this function might look and how it might sound. Have them predict how it is related to the original motif, both visually and musically. You might ask a student volunteer to sing the transformation.

Play the melody from the CD after all students have calculated the in/out table, converted the function values to notes, written the melody, and plotted its graph. It is important that students hear the variation (translation) in relation to the original melody so that the specific characteristics of adding a value to a function (translating it in the y direction) become clear.

As you play the function on the CD track, have students follow along the graph on their worksheets or trace the contour of the graph on the overhead in sync with the melody. Ask students to describe or write about the relationship between $f(x)$ and $f(x) + 2$, from both looking at the graph and listening to the melodies.

Mathematics perspective: Adding a value to the function has not changed the shape of the original, only its position on the coordinate grid. The geometric shape of $f(x) + 2$ is congruent to $f(x)$. The transformation can be thought of as either adding 2 to $f(x)$ to get $y = f(x) + 2$ or subtracting 2 from y to get $y - 2 = f(x)$. Thus we can make the observation that subtracting 2 from y moves the function *up* 2 units. (What might adding values to y do to the function?) The idea of values being added to and subtracted from x will be discussed in Functional Composer, Second Movement.

Music perspective: The melody $f(x) + 2$ is within a new scale, called a *mode* of the original scale. This melody in a different mode sounds similar in shape to the original melody, only higher in pitch. Technically the melody created here is not an exact translation of the original melody because the distances between adjacent notes in the major scale is not consistent throughout the scale. Thus the translated melodies will have slightly different qualities from the original, while the mathematical translations will be exactly the same. An exact translation of a melody is called a *transposition.* Turning a translation into a transposition requires adjusting the notes with flats and sharps. Discussing this subtlety only complicates matters and distracts from the message of the lesson, but you should be aware of it in case students bring it up.

Conclude by confirming that this transformation is a *translation* and summarize its characteristics: the congruence of the shapes and the direction of shift.

STEP 5 A second transformation: $y = 2f(x)$

TRACK 37

Ask the class to suggest another transformation that could be applied to $f(x)$ that might yield a characteristic different from that of the translation. Lead them to $y = 2f(x)$.

Students may suggest many transformations that will produce function values that cannot be converted to notes as defined by the system in Functional Composer. Function values must be integers, and any function value greater than 14 will be out of the musical range of available notes. Lead students to realize this if they suggest unusable transformations.

Present this and the following transformations using the format of Step 4. As students calculate, plot, and graph each function, play all previous transformations from the CD so that students hear the functions in relation to one another for comparison. Ask students to compare all previous functions to one another and to the original, both visually and musically.

Multiplying a function by a positive integer value stretches the function, changing its shape. It is no longer congruent to the original. What if you multiply the function by a fraction? In some cases this would yield fractional function values that cannot be graphed since $f(x)$ is only defined for integers. Ask students what relationship must exist between the function values and a fraction multiplier to create a graphable transformation. What about multiplying a function by a negative integer? Our fourth transformation will explore the effect of negative numbers.

Geometry connection: Make connections to the topics of geometric mapping: congruence mapping (isometries), reflections, and dilations. Identify which transformations are dilations—those resulting from multiplying the original function by something—as opposed to congruence mappings—those resulting from something being added to or subtracted from the original function.

Conclude by confirming that this transformation has the effect of stretching the original function in the vertical direction.

STEP 6 A third transformation: $y = 2f(x) + 4$

TRACK 38

Again, ask students for another possible way to alter the original function. A student might suggest doing both transformations: multiplying and adding. Lead students to $y = 2f(x) + 4$. Guess and discuss what this transformation will sound and look like and follow the same process as in previous transformations.

The topics of composite functions and/or order of operations can be explored here. What comes first: the stretching or the translation? Is stretching a function by a multiplier of 2 and then translating it by a value of 4 different from translating it by a value of 4 and then stretching it by a value of 2? What is the difference? How would you represent these two processes mathematically? Have students come up with an explanation and the mathematical expressions $y = 2(f(x) + 4)$ (translating before stretching) and $y = 2f(x) + 4$ (stretching before translating). Note that after simplifying, the expressions show a difference. Explain that $y = 2f(x) + 4$ is called a *composite function* because it is a combination of $y = 2f(x)$ and $y = f(x) + 4$. Review of this term can be used as a lead-in the next day for a lesson on composite functions.

Conclude by confirming that multiplying and then adding a value both translates and stretches the function.

STEP 7 A fourth transformation: $y = -f(x)$

TRACK 39

Now you can answer any inquiry about multiplying functions by negative numbers. Follow the same process as before as you present this transformation.

Notice that the transformation resulting from a negative multiplier creates a melody that may sound sinister. This can be a thought-provoking connection to make with students.

Conclude by confirming that multiplying a function by a negative value reflects the function across the x-axis.

STEP 8 Does this make music? A composition

TRACK 40

Refer back to our unhappy composer with writer's block. Ask students whether the melodies that have been created seem related to one another. Suggest that playing these melodies back to back and adding some accompaniment might be a way to create some real music that sounds good and "makes sense" to our ears.

Track number 40 on the CD is an example of this process. The melodies created from our transformations are assembled with some minimal accompaniment. Some melodies have been repeated and overlapped to create a simple example of how a musical composition might develop from our work. It is strikingly musical and supports our idea that mathematics is a big part of the structure that makes the music aesthetically pleasing.

Play the composition for students and discuss these aspects with them.

Have students identify the functions in the composition as it is played. You may need to play it several times. You might have students go to the overhead and trace the graph of a particular melody as it occurs in the composition.

FOLLOW-UP ACTIVITIES

Functional Composer, Second Movement

The Second Movement explores a question that arises in the First Movement: What is the effect of operating on the input value of the function? In the Second Movement, the original motif is repeated to become a periodic function, and the transformational and musical relationships are revealed in greater depth.

Name That Function

In this activity, students have the opportunity to write the algebraic expression for a mystery melody that is played for them. The activity can be continued throughout the year, perhaps as an occasional warm-up exercise to reinforce graphing concepts in a fun, gamelike format.

Writing prompts

It is valuable for students to reflect on what they have learned. Here are some good writing prompts:

- Did the music created by mathematics sound "good" to you? Explain why or why not.
- Do you think that the mathematics provided a good solution for the composer?
- Do you think that music that comes from inspiration is always better than music that is created mathematically? Explain why or why not.
- Under what circumstances do you think mathematics is an appropriate tool for writing music?
- How do you think mathematics relates to intuition and inspiration?

Textbook assignments

It is helpful to follow Functional Composer, First Movement with textbook assignments that use transformation of functions directly in a purely mathematical context. In this way the mathematical relevance of the activity is reinforced, and the experience of the activity will transfer more effectively to mathematical understanding.

Extensions

- Explore more original transformations and have students bring in musical instruments to play.
- Have students "compose" music algebraically by creating their own transformations and assembling them in different orders. Nonmusician students can create algebraic compositions, and their musician classmates can play them. Working in pairs as composer teams, they can make judgments about the musicality of each transformation, following a guess-and-check process to come up with something that sounds good to them. Is there some commonality within the families of transformations that sounds good? Why? This area is rich with possibilities for exploration.
- Consider including the element of rhythm and applying functions to rhythmic motifs.

ANSWERS

Melodic Variations and Transformations

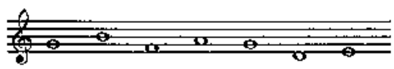
Original Motif

$y = f(x)$

3 5 2 4 3 0 1

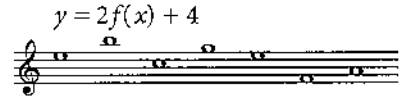
Melodic Variations

$y = f(x) + 2$



5 7 4 6 5 2 3

$y = 2f(x) + 4$



10 14 8 12 10 4 6

$y = 2f(x)$

6 10 4 8 6 0 2

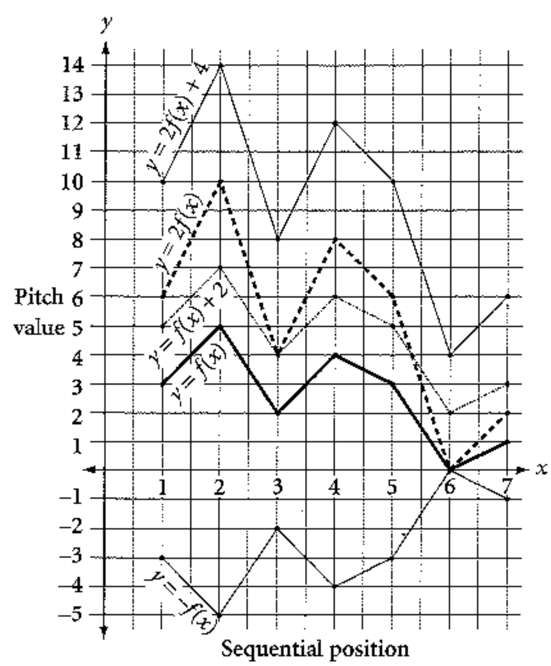
$y = -f(x)$

–3 –5 –2 –4 –3 0 –1

In/Out Table

x	$f(x)$	$f(x) + 2$	$2f(x)$	$2f(x) + 4$	$-f(x)$
1	3	5	6	10	–3
2	5	7	10	14	–5
3	2	4	4	8	–2
4	4	6	8	12	–4
5	3	5	6	10	–3
6	0	2	0	4	0
7	1	3	2	6	–1

Melodic-Contour Graph



TEACHER NOTES